EXERCISE – V HINTS & SOLUTIONS

Sol.1 (a) $\frac{a}{1-r} = 4 \Rightarrow a = 4 - 4r$

$ar = \frac{3}{4} \Rightarrow a = \frac{3}{4r} \qquad \Rightarrow 3 = 4r(4-4r)$

$\Rightarrow 16r^2 - 16r + 3 = 0 \Rightarrow (4r-3)(4r-1) = 0$

$\therefore r = \frac{3}{4}$ & $a = 1$ or $r = \frac{1}{4}$ & $a = 3$

(b) $a + b + c + d = 2$

$\frac{(a+b)+(c+d)}{2} \ge \sqrt{(a+b)(c+d)}$ {G.M.>0}

$\Rightarrow 1 \ge \sqrt{(a+b)(c+d)} \Rightarrow 1^2 \ge M \Rightarrow 0 < M \le 1$

M is greater than or equal to 0 & less than or equal to 1 or $0 \le M \le 1$

(c) $(a-3d)(a-d)(a+d)(a+3d) + (2d)^4$
$= (a^2 - 9d^2)(a^2 - d^2) + 16d^4$
$= a^4 - 10a^2d^2\ 9d^4 + 16d^4$
$= (a^2 - 5d^2)^2$
Here $(a-3d)$, $(a-d)$, $(a+d)$, $(a+3d)$ are integer
$\Rightarrow$ 2d is integer (but a & is not integer)
$= [(a^2 - d^2) - 4d^2]^2$
$= [(a-d)(a+d) - (2d)^2]^2$
$= [\text{integer} - \text{integer}]^2 \qquad = (\text{integer})^2$

Sol.2 $Ax^2 - 4x + 1 = 0 \langle^{\alpha}_{\gamma} \Rightarrow \alpha + \gamma = \frac{4}{A}, \alpha\gamma = \frac{1}{A}$

$Bx^2 - 6x + 1 = 0 \langle^{\beta}_{\delta} \Rightarrow \beta + \delta = \frac{6}{B}, \beta\delta = \frac{1}{B}$

$\alpha, \beta, \gamma, \delta$ in H.P.

$\beta = \frac{2\alpha\gamma}{\alpha+\gamma} = \frac{2.1}{4} \Rightarrow \beta = \frac{1}{2}$

& $\gamma = \frac{2\beta\delta}{\delta+\beta} = \frac{2.1}{6} \Rightarrow \gamma = \frac{1}{3}$

γ satisfy equation (1)

$A\left(\frac{1}{3}\right)^2 - 4 \times \frac{1}{3} + 1 = 0 \quad \Rightarrow A = 3$

β satisfy equation (ii)

$B\left(\frac{1}{2}\right)^2 - 6 \times \frac{1}{2} + 1 = 0$

$\Rightarrow B = 8$

Sol.3 $ax^2 + bx + c = 0 \langle^{\alpha}_{\beta}$

$\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} \Rightarrow (\alpha+\beta) = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$

$\Rightarrow (\alpha+\beta) = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$

$\Rightarrow -\frac{b}{a} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} \quad \Rightarrow \frac{-bc^2}{a^3} = \frac{b^2}{a^2} - \frac{2c}{a}$

$\Rightarrow -bc^2 = ab^2 - 2ca^2 \quad \Rightarrow 2ca^2 = ab^2 + bc^2$
$\Rightarrow bc^2, ca^2, ab^2$ in A.P.

Sol.4 $\log_2 x + \log_4 x + \log_{16} x + ... = y$

$\Rightarrow \log_2 x + \frac{1}{2}\log_2 x + \frac{1}{2^2}\log_2 x + ... = y$

$\Rightarrow (\log_2 x) \cdot \frac{1}{\left(1 - \frac{1}{2}\right)} = y$

$2\log_2 x = y \Rightarrow x^2 = 2^y$

& $\frac{5+9+13+...+(4y+1)}{1+3+5+...+(2y-1)} = 4\log_4 x$

$\Rightarrow \frac{\frac{y}{2}[5+4y+1]}{\frac{y}{2}[1+2y-1]} = 2\log_2 x$

$\Rightarrow \frac{2(2y+3)}{2y} = 2\log_2 x$

$\Rightarrow (2y+3) = 2y\left(\frac{y}{2}\right) \quad \Rightarrow 2y + 3 = y^2$

$\Rightarrow y^2 - 2y - 3 = 0 \qquad \Rightarrow (y-3)(y+1) = 0$
$\Rightarrow y = 3.\ y = -1$
not possible for II equation ($y \in N$)

$\Rightarrow \log_2 x = \frac{3}{2} \qquad \Rightarrow x = 2^{3/2} \Rightarrow x = 2\sqrt{2}$

Sol.5 (a) $x^2 - x + p = 0 \langle^{\alpha}_{\beta} \quad \alpha + \beta = 1, \alpha\beta = p$

$x^2 - 4x + q = 0 \langle^{\gamma}_{\delta} \quad \gamma + \delta = 4, \gamma\delta = q$

$\alpha, \beta, \gamma, \delta$ in G.P.
Let $\alpha = a$, $\beta = ar$, $\gamma = ar^2$, $\delta = ar^3$
$a + ar = 1$ & $ar^2 + ar^3 = 4$
$a(1 + r) = 1$ $\quad ar^2(1 + r) = 4$
$r^2 = 4 \Rightarrow r = \pm 2$

$r = +2 \Rightarrow a = \frac{1}{3} \Rightarrow \alpha = \frac{1}{3}, \beta = \frac{2}{3}, \gamma = \frac{4}{3}, \delta = \frac{8}{3}$
satisfy given equations but p & q not integer for r = 2
$\Rightarrow r \neq 2$
$\therefore r = -2$ $a = -1 \Rightarrow \alpha = -1, \beta = 2, \gamma = -4, \delta = 8$
$(-1)^2 - (-1) + p = 0 \Rightarrow p = -2$
& $(-4)^2 - 4(-4) + q = 0 \Rightarrow q = -32$

(b) $2 + 5 + 8 + ... + T_{2n} = 57 + 59 + 61 + ... + T_n$

$\Rightarrow \frac{2n}{2}[2.2 + (2n-1)3] = \frac{n}{2}[2.57 + (n-1)2]$
$\Rightarrow n[4 + 6n - 3] = n[57 + (n - 1)]$
$\Rightarrow 6n + 1 = n + 56$
$\Rightarrow 5n = 55 \Rightarrow n = 11$

(c) a, b, c, d in A.P.
$\Rightarrow$ d, c, b, a in A.P.

$\Rightarrow \frac{d}{abcd}, \frac{c}{abcd}, \frac{b}{abcd}, \frac{a}{abcd}$ in A.P.

$\Rightarrow \frac{1}{abc}, \frac{1}{abd}, \frac{1}{acd}, \frac{1}{bcd}$ in A.P.
$\Rightarrow$ abc, abd, acd, bcd in H.P.

(d) $a_1, a_2, a_3, ..., a_n$ (+) ve real numbers
$A_1 = a_1, G_1 = a_1, H_1 = a_1 \Rightarrow G_1 = (A_1H_1)^{1/2}$

$A_n = \frac{a_1 + a_2 + ... + a_n}{n}$, $G_n = (a_1a_2 ... a_n)^{1/n}$,

$H_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}} \Rightarrow G_n = (A_nH_n)^{1/2}$

G.M. of $G_1, G_2, G_3, ..., G_n$
$= (G_1, G_2, G_3, ..., G_n)^{1/n}$
$= [(A_1H_1)^{1/2}.(A_2H_2)^{1/2} ... (A_nH_n)^{1/2}]^{1/n}$
$= [(A_1 \cdot A_2 \cdot A_3 \cdot ..... A_n)(H_1 \cdot H_2 \cdot H_3 \cdot .... \cdot H_n)]^{1/2n}$

Sol.6 **(a)** $2b = a + c$ & $(b^2)^2 = a^2c^2$ & $a < b < c$

& $a + b + c = \frac{3}{2}$ & $b^2 = \pm ac$

$\Rightarrow 3b = \frac{3}{2}$

$b = \frac{1}{2}$ $\quad ac = \pm\frac{1}{4}$ & $a + c = 1$

roots of equation $x^2 - x \pm \frac{1}{4} = 0$

Case – I
$\Rightarrow 4x^2 - 4x + 1 = 0 \quad \Rightarrow (2x - 1)^2 = 0$
$\Rightarrow a = c$ (but $a < b < c$) $\quad \therefore$ reject

Case – II

$4x^2 - 4x - 1 = 0 \Rightarrow x = \frac{4 \pm 4\sqrt{2}}{8}$

$\Rightarrow x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$ $\qquad (\because a < b < c)$

$\Rightarrow a = \frac{1}{2} - \frac{1}{\sqrt{2}}$ & $c = \frac{1}{2} + \frac{1}{\sqrt{2}}$

(b) $A_1 + A_2 = a + b$ & $G_1G_2 = ab$

& $\frac{H_1 + H_2}{H_1H_2} = \frac{a + b}{ab}$

$\Rightarrow \frac{H_1 + H_2}{H_1H_2} = \frac{A_1 + A_2}{G_1G_2} \Rightarrow \frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2}$

$\frac{1}{H_1} = \frac{1}{a} + \frac{a - b}{3ab} \Rightarrow \frac{1}{H_1} = \frac{a + 2b}{3ab}$

$\Rightarrow \frac{1}{H_2} = \frac{1}{a} + \frac{2(a - b)}{3ab} \Rightarrow \frac{1}{H_2} = \frac{2a + b}{3ab}$

$\frac{G_1G_2}{H_1H_2} = ab \times \frac{(a + 2b)(2a + b)}{3ab \times 3ab}$

Sol.7 $2b = a + c$ & $b^2 = \frac{2a^2c^2}{a^2 + c^2}$

$4b^2 = a^2 + c^2 + 2ac \Rightarrow a^2 + c^2 = 4b^2 - 2ac$
& $b^2(a^2 + c^2) = 2a^2c^2$
$b^2(4b^2 - 2ac) = 2a^2c^2$
$\Rightarrow b^2(2b^2 - ac) = a^2c^2$
$\Rightarrow 2b^4 - b^2(ac) - (ac)^2 = 0$
$\Rightarrow (b^2 - ac)(2b^2 + ac) = 0$
If $b^2 = ac$ or $2b^2 = -ac$

$\Rightarrow$ a, b, c in G.P. $\qquad b^2 = \left(\frac{-c}{2}\right) \times a$

& a, b, c in A.P. also $\qquad a, b, \frac{-c}{2}$ in G.P.

$\Rightarrow a = b = c$

Sol.8 $S_\infty = \frac{a}{1 - r} = 5$

$\Rightarrow 5 - 5r = x \qquad \Rightarrow 5r = 5 - x$
$\Rightarrow |r| < 1 \qquad \Rightarrow -5 < 5r < +5$
$\Rightarrow -5 < x - 5 < 5 \qquad \Rightarrow 0 < x < 10$

Sol.9 $(1 + a) \geq 2\sqrt{a}$

$\{(1 + a)(1 + b)(1 + c) = 1 + (a + b + c + ab + bc + ca + abc)$

$(1 + b) \geq 2\sqrt{b}$

$(1 + c) \geq 2\sqrt{c}$

$(1 + a)(1 + b)(1 + c) \geq 8(abc)^{1/2}$

$[(1 + a)(1 + b)(1 + c)]^8 \geq 8^8 (abc)^4$

$[(1 + a)(1 + b)(1 + c)]^7 > 7^7 (a^4.b^4.c^4)$

equally holds a = b = c = 1

$\therefore 8^8 > 7^7$

Aliter

$$\frac{a+b+c+ab+bc+ca+abc}{7} \geq (a^4b^4.c^4)^{1/7}$$

$[(1 + a)(1 + b)(1 + c) - 1] \geq 7(a^4.b^4.c^4)^{1/7}$

$\Rightarrow (1 + a)(1 + b)(1 + c) > 7(a^4 b^4 c^4)^{1/7}$

$\Rightarrow [(1 + a)(1 + b)(1 + c)]^7 > 7^7 (a^4 . b^4 . c^4)$

Sol.10 (a) $ax^2 + bx + c = 0 \begin{matrix} \alpha \\ \beta \end{matrix}$

$\Delta = b^2 - 4ac$, quadratic $\Rightarrow a \neq 0$

$\alpha + \beta, \alpha^2 + \beta^2, \alpha^3 + \beta^3$ in G.P.

$(\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$

$\Rightarrow \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = \alpha^4 + \beta^4 + \alpha\beta(\alpha^2 + \beta^2)$

$\Rightarrow \alpha\beta(\alpha^2 + \beta^2 - 2\alpha\beta) = 0$

$\Rightarrow \alpha\beta(\alpha - \beta)^2 = 0$

$$\Rightarrow \frac{c}{a}\ \frac{(b^2 - 4ac)}{a^2} = 0 \quad \Rightarrow c\Delta = 0$$

(b) $S_n = \left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$

$T_k = k.2^{n+1-k}$

$$S_n = \sum_{k=1}^{n} k.2^{n+1-k} = 2^{n+1}\sum_{k=1}^{n} k.2^{-k}$$

$S_n = 1.2^n + 2.2^{n-1} + 3.2^{n-2} + ... + n.2$

$$\frac{S_n}{2} = \quad 1.2^{n-1} + 2.2^{n-2} + ... + (n-1).2 + n$$

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$$\frac{S_n}{2} = (2^n + 2^{n-1} + 2^{n-2} + ... + 2) - n$$

$$\frac{S_n}{2} = \frac{2(2^n - 1)}{2 - 1} - n$$

$$S_n = 2[2^{n+1} - n - 2] = \left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$$

$$\because \{(2^{n+1} - n - 2)\} \neq 0 \quad \Rightarrow 2 = \frac{n+1}{4} \Rightarrow n = 7$$

Sol.11 $A_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + ... + (-1)^{n-1}\left(\frac{3}{4}\right)^n$

$$A_n = \frac{\frac{3}{4}\left[1 - \left(-\frac{3}{4}\right)^n\right]}{1 - \left(\frac{-3}{4}\right)} = \frac{3}{7}\left[1 - \left(\frac{-3}{4}\right)^n\right]$$

$B_n = 1 - A_n$ & $B_n > A_n$

$$\Rightarrow 2A_n < 1 \qquad \Rightarrow \frac{6}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right) < 1$$

$$\Rightarrow 1 - \left(-\frac{3}{4}\right)^n < \frac{7}{6} \Rightarrow -\frac{1}{6} < \left(-\frac{3}{4}\right)^n$$

$\Rightarrow 2^{2n-1} > (-3)^{n+1}$

satisfy for n = 6, 7, 8 ....

which is greater 5 = n_0

Sol.12 (a) $V_r = \frac{r}{2}[2r + (r - 1)(2r - 1)] = \frac{1}{2}[2r^3 - r^2 + r]$

$$\Sigma V_r = \frac{1}{12} n(n + 1)(3n^2 + n + 2)$$

(b) $V_{r+1} - V_r = (r + 1)^3 - r^3 - \frac{1}{2}[(r + 1)^2 - r^2] + \frac{1}{2}(1)$

$\Rightarrow T_r = 3r^2 + 2r - 1 \quad \Rightarrow T_r = (r + 1)(3r - 1)$

which is composite no.

(c) $T_r = 3r^2 + 2r - 1$

$T_{r+1} = 3(r + 1)^2 + 2(r + 1) - 1$

$Q_r = T_{r+1} - T_r = 3(2r + 1) + 2(1)$

$Q_r = 6r + 5$

$Q_{r+1} = 6(r + 1) + 5$

common difference = $Q_{r+1} - Q_r = 6$

Sol.13 (a) $A_1 = \frac{a+b}{2}$, $G_1 = \sqrt{ab}$, $H_1 = \frac{2ab}{a+b}$

$A_n = \frac{A_{n-1} + H_{n-1}}{2}$, $G_n = \sqrt{A_{n-1} H_{n-1}}$,

$$H_n = \frac{2A_{n-1} H_{n-1}}{A_{n-1} + H_{n-1}}$$

$G_1 = G_2 = G_3 = ...= Gn = \sqrt{ab}$

(b) A_2 is A.M. of A_1 and H_1

& $A_1 > H_1 \Rightarrow A_1 > A_2 > H_1$

A_3 is A.M. of A_2 and H_2 & $A_2 > H_2 \Rightarrow A_2 > A_3 > H_2$

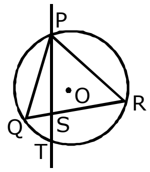
$\therefore A_1 > A_2 > A_3 >$ ....

(c) As above $A_1 > H_2 > H_1$ & $A_2 > H_3 > H_2$

$\therefore H_1 < H_2 < H_3 <$ ....

Sol.14 (a) (PS) × (ST) = (QS) × (SR)

$$\frac{\left(\frac{1}{PS}\right)+\left(\frac{1}{ST}\right)}{2} > \sqrt{\frac{1}{PS}\times\frac{1}{ST}}$$



$$\Rightarrow \frac{1}{PS} + \frac{1}{ST} > \frac{2}{\sqrt{QS \times SR}} \quad .....(i)$$

$$\frac{QS+SR}{2} \Rightarrow \sqrt{QS\times SR} \Rightarrow \frac{1}{\sqrt{QS\times SR}} > \frac{2}{QR} \quad (ii)$$

from (i) & (ii) $\frac{1}{PS} + \frac{1}{ST} > \frac{4}{QR}$

(b) $b_1 = a_1$
$b_2 = a_1 + a_2$
$b_3 = a_1 + a_2 + a_3$
$b_4 = a_1 + a_2 + a_3 + a_4$
Hence b_1, b_2, b_3, b_4 are neither in A.P. nor in G.P. nor in H.P.

Sol.15 $t_n = c\,\{n^2 - (n-1)\}^2$
$= c\,(2n - 1)$
$\Rightarrow t^2_n = c^2\,(4n^2 - 4n + 1)$

$$\Rightarrow \sum_{n=1}^{n} t_n^2 = c^2 \left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{6}\,\{4(n+1)(2n+1) - 12(n+1) + 6\}$$

$$= \frac{c^2 n}{3}\,\{4n^2 + 6n + 2 - 6n - 6 + 3\} = \frac{c^2 n}{3}\,(4n^2 - 1)$$

Sol.16 $S_k = \dfrac{\frac{k-1}{k!}}{1-\frac{1}{k}} = \dfrac{1}{(k-1)!}$ but $k \neq 1$

$$\sum_{k=2}^{100}\left|(k^2-3k+1)\frac{1}{(k-1)!}\right| = \sum_{k=2}^{100}\left|\frac{(k-1)^2-k}{(k-1)!}\right|$$

$$= \sum\left|\frac{k-1}{(k-2)!} - \frac{k}{(k-1)!}\right|$$

$$= \left|\frac{1}{0!}-\frac{1}{1!}\right| + \left|\frac{2}{1!}-\frac{3}{2!}\right| + \left|\frac{3}{2!}-\frac{4}{3!}\right| + ... + \left|\frac{99}{98!}-\frac{100}{99!}\right|$$

$$= -1 + 2 + 2 - \frac{3}{2!} + \frac{3}{2!} - \frac{4}{3!} + ... + \frac{99}{98!} - \frac{100}{99!}$$

$$= 3 - \frac{100}{99!}$$

Sol.17 $a_k = 2a_{k-1} - a_{k-2}$
$\Rightarrow a_1, a_2, a_3, ..., a_{11}$ are in A.P. with common difference d.
$a_1 = 15,\ a_2 = 15 + d,\ a_3 = 15 + 2d, ...,\ a_{11} = 15 + 10d$

$$\frac{a_1^2 + a_2^2 + a_3^2 + ... + a_{11}^2}{11}$$

$$= \frac{15^2\cdot 11 + 30d(11\cdot 5) + d^2\,35\cdot(11)}{11} = 90$$

$\Rightarrow 35d^2 + 150d + 225 = 90$
$\Rightarrow 35d^2 + 150d + 135 = 0$
$\Rightarrow 7d^2 + 30d + 27 = 0$

$$\Rightarrow d = -3,\ -\frac{9}{7} \quad \because\ a_2 < \frac{27}{2} \quad \therefore\ d \neq -\frac{9}{7}$$

$$\therefore \frac{a_1 + a_2 + a_3 + ... + a_{11}}{11} = \frac{11}{2}\,\frac{[30 + 10(-3)]}{11} = 0$$

18. 0008
Using AM $\geq$ GM

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^{8} + a^{10} + 1}{8} \geq 1$$

$\Rightarrow$ Sum = 8

19. 0009

$$\frac{s_m}{s_n} = \frac{\frac{m}{2}\{2a_1 + (m-1)d\}}{\frac{n}{2}\{2a_1 + (n-1)d\}} = \frac{\frac{5n}{2}\{2a_1 + (5n-1)\,d\}}{\frac{n}{2}\{2a_1 + (n-1)\,d\}}$$

$2a_1 - d = 0 \Rightarrow d = 2a_1$
$\Rightarrow d = 2a_1 = 6 \Rightarrow a_2 = 9$

20. D
$a_1 = 5,\ a_{20} = 25$

$$t_1 = \frac{1}{5},\ t_{20} = \frac{1}{25} = t_1 + 19.d$$

$$\Rightarrow \frac{1}{25} = \frac{1}{5} + 19.d \Rightarrow -\frac{4}{25.19} = d$$

$$\Rightarrow \frac{1}{5} - \frac{(n-1)4}{25.19} < 0 \quad \Rightarrow \frac{(n-1)4}{25.19} > \frac{1}{5}$$

$$\Rightarrow (n-1) > \frac{19\cdot 5}{4} \Rightarrow n > 24.7$$